

ON CONNECTIONS OF GENERALIZED ENTROPIES WITH SHANNON AND KOLMOGOROV-SINAI ENTROPIES

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ABSTRACT. We consider the concept of generalized measure-theoretic entropy, where instead of the Shannon entropy function we consider an arbitrary concave function defined on the unit interval, vanishing in the origin. Under mild assumptions on this function we show that this isomorphism invariant is linearly dependent on the Kolmogorov-Sinai entropy.

1. INTRODUCTION

Dynamical and measure-theoretic (called also Kolmogorov-Sinai entropy) entropies are a basic tool for investigating dynamical systems (see e.g. [5, 9]). They were extensively studied and successfully applied among others in statistical physics and quantum information. It appeared to be an exceptionally powerful tool for exploring nonlinear systems. One of the biggest advantages of the Kolmogorov-Sinai entropies lies in the fact that it makes possible to distinguish the formally regular systems (those with the measure-theoretic entropy equal to zero) from the chaotic ones (with positive entropy, which implies positivity of topological entropy [11]).

The Kolmogorov-Sinai entropy of a given transformation T acting on a probability space (X, Σ, μ) is defined as the supremum over all finite measurable partitions \mathcal{P} of the dynamical entropy of T with respect to \mathcal{P} , denoted by $h(T, \mathcal{P})$. As a dynamical counterpart of Shannon entropy, the entropy of transformation T with respect to a given partition \mathcal{P} is defined as the limit of the sequence $(\frac{1}{n}H(\mathcal{P}_n))_{n=1}^{\infty}$, where

$$H(\mathcal{P}_n) = \sum_{A \in \mathcal{P}_n} \eta(\mu(A))$$

with η being the Shannon function given by $\eta(x) = -x \log x$ for $x > 0$ with $\eta(0) := 0$ and \mathcal{P}_n is the join partition of partitions $T^{-i}\mathcal{P}$ for $i = 0, \dots, n-1$. The existence of the limit in the definition of the dynamical entropy follows from the subadditivity of η . The most common interpretation of this quantity is the average (over time and the phase space) one-step gain of information about the initial state. Taking supremum over all finite partitions we obtain an isomorphism invariant which measures the rate of producing randomness (chaos) by the system.

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Since Shannon's seminal paper [20] many generalizations of the concept of Shannon static entropy were considered, see Arimoto [1], Rényi [17] and Csiszár's survey article [4]. The dynamical and measure-theoretic counterparts were considered by few authors. De Paly [15] proposed generalized dynamical entropies based on the concept of the relative static entropies. Unfortunately it appeared that, despite some special cases [15, 16] the explicit calculations of this invariant may not be possible. Grassberger and Procaccia proposed in [7] a dynamical counterpart of the well-known generalization of Shannon entropy – the Rényi entropy, and its measure-theoretic counterpart were considered by Takens and Verbitski. They showed that for ergodic transformations with positive measure-theoretic entropy, Rényi entropies of a measure-theoretic transformation are either infinite or equal to the measure-theoretic entropy [22]. The answer for non-ergodic aperiodic transformations is different, for Rényi entropies of order $\alpha > 1$ they are equal to the essential infimum of the measure-theoretic entropies of measures forming the decomposition of a given measure into ergodic components, while for $\alpha < 1$ they are still infinite [23]. In particular, this means that Rényi entropies of order $\alpha < 1$ are metric invariants sensitive to ergodicity. Similar generalization was made by Mesón and Vericat [12, 13] for so called Havrda-Charvát-Tsallis entropy [8] and their results were similar to ones obtained by Takens and Verbitski in [22].

In our approach is based on Arimoto generalization applied to dynamical case. Instead of the Shannon function η we consider a concave function $g: [0, 1] \mapsto \mathbb{R}$ such that $\lim_{x \rightarrow 0^+} g(x) = g(0) = 0$ and define the dynamical g -entropy of the finite partition \mathcal{P} as

$$h(g, T, \mathcal{P}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{A \in \mathcal{P}_n} g(\mu(A)).$$

The behaviour of the quotient $g(x)/\eta(x)$ as x converges to zero appears to be crucial for our considerations. Mainly, defining

$$\text{Ci}(g) := \liminf_{x \rightarrow 0^+} \frac{g(x)}{\eta(x)} \quad \text{and} \quad \text{Cs}(g) := \limsup_{x \rightarrow 0^+} \frac{g(x)}{\eta(x)}$$

we will prove that

$$\text{Ci}(g) \cdot h(T, \mathcal{P}) \leq h(g, T, \mathcal{P}) \leq \text{Cs}(g) \cdot h(T, \mathcal{P}).$$

Moreover for g such that $\text{Ci}(g) = \infty$, which fullfills some additional condition (e. g. for $g(x) = \sqrt{x}$), we can find zero dynamical entropy processes $(X, \Sigma, \mu, T, \mathcal{P})$ with a given positive dynamical g -entropy.

Taking the supremum over all partitions we obtain Kolmogorov entropy-like isomorphism invariant, which we will call the measure-theoretic g -entropy of a transformation with respect to an invariant measure. One might ask whether this

invariant may give any new information about the system. We will prove (Theorem 4.2) that for g with $\text{Cs}(g) < \infty$, this new invariant is linearly dependent on Kolmogorov-Sinai entropy. It means that in fact the Shannon entropy function is the most natural one – not only it has all of the properties which the entropy function should have [5], but also considering different entropy functions we will not obtain essentially different invariant. This result might have the other interpretation. Ornstein and Weiss showed in [14] that every finitely observable invariant for the class of all ergodic processes has to be a continuous function of the entropy. It is easy to see that any continuous function of the entropy is finitely observable – one simply composes the entropy estimators with the continuous function itself. In other words an isomorphism invariant is finitely observable if and only if it is a continuous function of the Kolmogorov-Sinai entropy. Therefore our result implies that the generalized measure-theoretic entropy is in fact finitely observable. It should be possible to give a more direct proof of the finite observability of the generalized measure-theoretic entropy but the proof cannot be easier¹ than the proof that entropy itself is finitely observable, see [24].

The paper is organized as follows: in the next section we give a formal definition of the dynamical g -entropy and establish its basic properties. The subsequent section is devoted to the construction of a zero dynamical entropy process with a given positive g -entropy. Finally, in the last section, we define a measure-theoretic g -entropy of a transformation and show connections between this new invariant and the Kolmogorov-Sinai entropy.

2. BASIC FACTS AND DEFINITIONS

Let (X, Σ, μ) be a Lebesgue space and let $g : [0, 1] \mapsto \mathbb{R}$ be a concave function with $g(0) = \lim_{x \rightarrow 0^+} g(x) = 0$.² By \mathcal{G}_0 we will denote the set of all such functions. Every $g \in \mathcal{G}_0$ is subadditive, i. e. $g(x + y) \leq g(x) + g(y)$ for every $x, y \in [0, 1]$, and quasihomogenic, i.e. $\varphi_g : (0, 1] \rightarrow \mathbb{R}$ defined by $\varphi_g(x) := g(x)/x$ is decreasing (see [19]).³

For a given finite partition \mathcal{P} we define the g -entropy of the partition \mathcal{P} as

$$(1) \quad H(g, \mathcal{P}) := \sum_{A \in \mathcal{P}} g(\mu(A)).$$

For $g = \eta$ the latter is equal to the Shannon entropy of the partition \mathcal{P} . For two finite partitions \mathcal{P} and \mathcal{Q} of the space X we define a new partition $\mathcal{P} \vee \mathcal{Q}$ (join

¹Benjamin Weiss personal communication

²We might assume only that $g(0) = 0$, but then the idea of the dynamical g -entropy would fail, since if $\mathcal{P}_{n+1} \neq \mathcal{P}_n$ for every n and $\lim_{x \rightarrow 0^+} g(x) \neq 0$, then the dynamical g -entropy of the partition \mathcal{P} would be infinite. Therefore, if g is not well-defined at zero we will assume that $g(0) := \lim_{x \rightarrow 0^+} g(x)$.

³If g is fixed we will omit the index, writing just φ .

partition of \mathcal{P} and \mathcal{Q}) consisting of the subsets of the form $B \cap C$ where $B \in \mathcal{P}$ and $C \in \mathcal{Q}$. The join partition of more than two partitions is defined similarly.

2.1. Dynamical g -entropies. For an automorphism $T: X \mapsto X$ and a partition $\mathcal{P} = \{E_1, \dots, E_k\}$ we put

$$T^{-j}\mathcal{P} := \{T^{-j}E_1, \dots, T^{-j}E_k\}$$

and

$$\mathcal{P}_n = \mathcal{P} \vee T^{-1}\mathcal{P} \vee \dots \vee T^{-n+1}\mathcal{P}.$$

Now for a given $g \in \mathcal{G}_0$ and a finite partition \mathcal{P} we can define the **dynamical g -entropy** of the transformation T with respect to \mathcal{P} as

$$(2) \quad h_\mu(g, T, \mathcal{P}) = \limsup_{n \rightarrow \infty} \frac{1}{n} H(g, \mathcal{P}_n).$$

Alternatively we will call it the g -entropy of the process $(X, \Sigma, \mu, T, \mathcal{P})$. If the dynamical system (X, Σ, T, μ) is fixed then we omit T , writing just $h(g, \mathcal{P})$. As in the case of Shannon dynamical entropies we are interested in the existence of the limit of $(\frac{1}{n}H(g, \mathcal{P}_n))_{n=1}^\infty$. If $g = \eta$, we obtain the Shannon dynamical entropy $h(T, \mathcal{P})$. However, in the general case we can not replace an upper limit in (2) by the limit, since it might not exist. Existence of the limit in the case of the Shannon function follows from the subadditivity of the static Shannon entropy. This property has every subderivative function, i.e. function for which the inequality $g(xy) \leq xg(y) + yg(x)$ holds for any $x, y \in [0, 1]$, but this is not true in general (an appropriate example will be given in Section 2.2). Therefore we propose more general classes of functions for which the limit exists. It exists if g belongs to one of two following classes:

$$\mathcal{G}_0^0 := \left\{ g \in \mathcal{G}_0 \mid \lim_{x \rightarrow 0^+} \frac{g(x)}{\eta(x)} = 0 \right\} \quad \text{or} \quad \mathcal{G}_0^{\text{Sh}} := \left\{ g \in \mathcal{G}_0 \mid 0 < \lim_{x \rightarrow 0^+} \frac{g(x)}{\eta(x)} < \infty \right\}.$$

It is easy to show that if g is subderivative then the limit $\lim_{x \rightarrow 0^+} g(x)/\eta(x)$ is finite. Moreover we will see that values of dynamical g -entropies depend on the behaviour of g in the neighbourhood of zero. We will prove that if $g \in \mathcal{G}_0^0 \cup \mathcal{G}_0^{\text{Sh}}$, then there is a linear dependence between the dynamical g -entropy and the Shannon dynamical entropy of a given partition. First we give the following general result:

Theorem 2.1. *If $g_1, g_2 \in \mathcal{G}_0$ are such that*

$$\liminf_{x \rightarrow 0^+} \frac{g_1(x)}{g_2(x)} < \infty,$$

and \mathcal{P} is a finite partition of X with finite dynamical g_2 -entropy, then

$$\liminf_{x \rightarrow 0^+} \frac{g_1(x)}{g_2(x)} \cdot h(g_2, \mathcal{P}) \leq h(g_1, \mathcal{P}).$$

If additionally $\limsup_{x \rightarrow 0^+} \frac{g_1(x)}{g_2(x)} < \infty$, then

$$h(g_1, \mathcal{P}) \leq \limsup_{x \rightarrow 0^+} \frac{g_1(x)}{g_2(x)} \cdot h(g_2, \mathcal{P}).$$

Remark 2.1. Whenever $g_2: [0, 1] \mapsto \mathbb{R}$ is a nonnegative concave function satisfying $g_2(0) = 0$ and $g_2'(0) = \infty$, we can have any pair $0 < a \leq b \leq \infty$ as limit inferior and limit superior of g_1/g_2 in 0, choosing a suitable function g_1 . The idea is as follows: construct g_1 piecewise linear. To do so define inductively a strictly decreasing sequence $x_k \rightarrow 0$, and a decreasing sequence of values $y_k = g_1(x_k) \rightarrow 0$, thus defining intervals $J_k := [x_{k+1}, x_k]$ where g is affine. The only constraint to get a concave function is that the slope of g on each interval J_k has to be smaller than y_k/x_k , and increasing with respect to k ; this is not an obstruction to approach any limit inferior and limit superior for $g_1(x)/g_2(x)$, provided that $x_{k+1} > 0$ is chosen small enough.

Proof of Theorem 2.1. Let $g_1, g_2 \in \mathcal{G}_0^0$ and assume that

$$\limsup_{x \rightarrow 0^+} \frac{g_1(x)}{g_2(x)} < \infty.$$

If the limit $\lim_{n \rightarrow \infty} H(g_2, \mathcal{P}_n)$ is finite, then $h(g_2, \mathcal{P}) = 0$ and

$$0 \leq h(g_1, \mathcal{P}) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \frac{H(g_1, \mathcal{P}_n)}{H(g_2, \mathcal{P}_n)} H(g_2, \mathcal{P}_n) \leq \limsup_{x \rightarrow 0^+} \frac{g_1(x)}{g_2(x)} \cdot \limsup_{n \rightarrow \infty} \frac{1}{n} H(g_2, \mathcal{P}_n) = 0$$

so we can assume that it is infinite and that $\lim_{x \rightarrow 0^+} \varphi_{g_1}(x) = \infty$ is also infinite, since $H(g, \mathcal{Q}) < \lim_{x \rightarrow 0^+} \varphi_{g_1}(x)$ for any partition \mathcal{Q} .

Fix $\varepsilon > 0$ and $\delta > 0$ such that for $x \in (0, \delta]$ we have

$$\liminf_{x \rightarrow 0^+} \frac{g_1(x)}{g_2(x)} - \varepsilon < \frac{g_1(x)}{g_2(x)} \leq \limsup_{x \rightarrow 0^+} \frac{g_1(x)}{g_2(x)} + \varepsilon.$$

Then for every index n we have

$$\frac{1}{\delta} \overline{G_\delta} \leq \sum_{B \in \mathcal{P}_n, \mu(B) \geq \delta} g_1(\mu(B)) \leq \frac{1}{\delta} \overline{G_\delta}.$$

where $\overline{G_\delta} := \max_{x \in [\delta, 1]} g_1(x)$, $\underline{G_\delta} = \min_{x \in [\delta, 1]} g_1(x)$, and

$$\liminf_{x \rightarrow 0^+} \frac{g_1(x)}{g_2(x)} - \varepsilon \leq \sum_{B \in \mathcal{P}_n, \mu(B) < \delta} g_1(\mu(B)) \bigg/ \sum_{B \in \mathcal{P}_n, \mu(B) < \delta} g_2(\mu(B)) \leq \limsup_{x \rightarrow 0^+} \frac{g_1(x)}{g_2(x)} + \varepsilon.$$

Therefore

$$\begin{aligned}
\liminf_{x \rightarrow 0^+} \frac{g_1(x)}{g_2(x)} - \varepsilon &\leq \frac{\sum_{\mu(B) < \delta} g_1(\mu(B)) \bigg/ \sum_{\mu(B) < \delta} g_2(\mu(B)) + \sum_{\mu(B) \geq \delta} g_1(\mu(B)) \bigg/ \sum_{\mu(B) < \delta} g_2(\mu(B))}{1 + \sum_{\mu(B) \geq \delta} g_2(\mu(B)) \bigg/ \sum_{\mu(B) < \delta} g_2(\mu(B))} \\
&\leq \limsup_{x \rightarrow 0^+} \frac{g_1(x)}{g_2(x)} + \varepsilon.
\end{aligned}$$

Converging with n to infinity and with ε and δ to zero succesively we obtain the assertion. In the case of infinite limit superior of the quotient $g_1(x)/g_2(x)$ we can repeat the above reasoning just omitting an upper bound for considered expressions. \square

Theorem 2.1 immediately implies few corollaries

Corollary 2.1. *If the limit $\lim_{x \rightarrow 0^+} \frac{g_1(x)}{g_2(x)}$ exists and is finite, then*

$$h(g_1, \mathcal{P}) = \lim_{x \rightarrow 0^+} \frac{g_1(x)}{g_2(x)} \cdot h(g_2, \mathcal{P}).$$

Corollary 2.2. *If $g \in \mathcal{G}_0^0 \cup \mathcal{G}_0^{\text{Sh}}$, then*

$$h(g, \mathcal{P}) = C(g) \cdot h(\mathcal{P}),$$

where $C(g) = \lim_{x \rightarrow 0^+} \frac{g(x)}{\eta(x)}$.

Corollary 2.3. *If $g \in \mathcal{G}_0^0 \cup \mathcal{G}_0^{\text{Sh}}$, then $h(g, \mathcal{P}) = \lim_{n \rightarrow \infty} \frac{1}{n} H(g, \mathcal{P}_n)$.*

Moreover using similar arguments we might obtain the answer in the case of infinite limit $\lim_{x \rightarrow 0^+} g_1(x)/g_2(x)$ and positive dynamical g_2 -entropy:

Proposition 2.1. *Let $g_1, g_2 \in \mathcal{G}_0$ be such that $\lim_{x \rightarrow 0^+} g_1(x)/g_2(x) = \infty$ and let a finite partition \mathcal{P} has positive g_2 -entropy. Then $h(g_1, \mathcal{P})$ is infinite.*

Let us define

$$\mathcal{G}_0^\infty := \left\{ g \in \mathcal{G}_0 \mid \lim_{x \rightarrow 0^+} \frac{g(x)}{\eta(x)} = \infty \right\},$$

then we can formulate the following fact:

Corollary 2.4. *If $g \in \mathcal{G}_0^\infty$ and a partition \mathcal{P} has positive Shannon dynamical entropy, then $h(g, \mathcal{P})$ is infinite.*

2.2. Bernoulli shifts. Let $\mathcal{A} = \{1, \dots, k\}$ be a finite alphabet. Let $X = \{x = \{x_i\}_{i=-\infty}^\infty : x_i \in \mathcal{A}\}$ and σ be a left shift

$$\sigma(x)_i = x_{i+1}.$$

For any $s \leq t$ and block $[\omega_0, \dots, \omega_{t-s}]$ with $a_i \in \mathcal{A}$ we define a cylinder

$$C_s^t(\omega_0, \dots, \omega_{t-s}) = \{x \in X : x_i = \omega_{i-s} \text{ for } i = s, \dots, t\}.$$

We consider the Borel σ -algebra with respect to the metric, which is given by $d(x, y) = 2^{-N}$, where $N = \min\{|i| : x_i \neq y_i\}$. One can show that Borel σ -algebra is the minimal σ -algebra containing all cylindrical sets. Let $p = (p_1, \dots, p_k)$ be a probability vector, i.e. $p_i \geq 0$ for any i and $\sum p_i = 1$. We define a measure $\rho = \rho(p)$ on \mathcal{A} by setting $\rho(\{i\}) = p_i$. Then μ_p is a corresponding product measure on $X = \mathcal{A}^{\mathbb{Z}}$. Thus, the static g -entropy of a partition $\mathcal{P}^{\mathcal{A}} = \{[1], [2], \dots, [k]\}$ is equal to

$$H_{\mu_p}(g, \mathcal{P}_n^{\mathcal{A}}) = \sum_{[\omega_0, \dots, \omega_{n-1}] \in \mathcal{A}^n} g(C_0^{n-1}(\omega_0, \dots, \omega_{n-1})) = \sum_{[\omega_0, \dots, \omega_{n-1}] \in \mathcal{A}^n} g(p_{\omega_0} \cdots p_{\omega_{n-1}}).$$

By the concavity of the function g we have

$$H_{\mu_p}(g, \mathcal{P}_n^{\mathcal{A}}) \leq \varphi\left(\frac{1}{k^n}\right)$$

where equality holds only when $p = p^* = (\frac{1}{k}, \dots, \frac{1}{k})$. Before calculating the dynamical g -entropy of the partition $\mathcal{P}^{\mathcal{A}}$ with respect to measure μ_{p^*} , we give the following lemma, which proof will be given later:

Lemma 2.1. *If $g \in \mathcal{G}_0$, then*

$$\text{Cs}(g) = \limsup_{n \rightarrow \infty} \frac{g(k^{-n})}{\eta(k^{-n})} \quad \text{and} \quad \text{Ci}(g) = \liminf_{n \rightarrow \infty} \frac{g(k^{-n})}{\eta(k^{-n})}$$

for any positive integer $k > 1$.

Therefore, applying Lemma 2.1 for the partition $\mathcal{P}^{\mathcal{A}}$ we obtain

$$(3) \quad h_{\mu_{p^*}}(g, \mathcal{P}^{\mathcal{A}}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \varphi\left(\frac{1}{k^n}\right) = \begin{cases} \text{Cs}(g) \cdot \log k, & \text{if } \text{Cs}(g) < \infty; \\ \infty, & \text{otherwise.} \end{cases}$$

Remark 2.2. *If we consider lower limit instead of the upper limit we would obtain*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \varphi\left(\frac{1}{k^n}\right) = \begin{cases} \text{Ci}(g) \cdot \log k, & \text{if } \text{Ci}(g) < \infty; \\ \infty, & \text{otherwise.} \end{cases}$$

Therefore we can not replace an upper limit by the limit in the definition of the dynamical g -entropy.

Proof of Lemma 2.1. We will show the equality for the upper limit. Proof of the equality for the lower limit is similar. Let $(x_n)_{n=1}^{\infty}$ and $(m_n)_{n=1}^{\infty}$ be such that $\limsup_{n \rightarrow \infty} g(x_n)/\eta(x_n) = c$ and $x_n \in (k^{-m_n}, k^{-m_n+1})$ for every $n \in \mathbb{N}$. Then $-\log x_n \geq -\log k^{-m_n+1}$. Every function $g \in \mathcal{G}_0$ is quasihomogenic, so for every

positive integer n occurs

$$\frac{g(x_n)}{x_n} < \frac{g(k^{-m_n})}{k^{-m_n}}.$$

Therefore

$$\begin{aligned} \frac{g(x_n)}{\eta(x_n)} &= \frac{g(x_n)}{x_n} \frac{1}{-\log x_n} \leq \frac{g(k^{-m_n})}{k^{-m_n}} \frac{1}{(m_n - 1) \log k} \\ &= \frac{g(k^{-m_n})}{\eta(k^{-m_n})} \cdot \frac{m_n}{m_n - 1}, \end{aligned}$$

and

$$\limsup_{x \rightarrow 0^+} \frac{g(x)}{\eta(x)} = \limsup_{n \rightarrow \infty} \frac{g(k^{-n})}{\eta(k^{-n})}.$$

□

3. ZERO DYNAMICAL ENTROPY PROCESSES WITH g -ENTROPY OF A GIVEN VALUE

As we have seen in the previous section in the case of positive Shannon dynamical entropy and $g \in \mathcal{G}_0^\infty$ the dynamical g -entropy is always infinite. The case when the Shannon dynamical entropy is equal zero is different. In this section we will prove that for $g \in \mathcal{G}_0^\infty$ which fullfill some additional assumptions, i.e.

$$\lim_{\lambda \rightarrow \infty} \liminf_{x \rightarrow 0^+} \frac{\lambda g(x)}{g(\lambda x)} > 1,$$

there exist processes of a given dynamical g -entropy and zero Shannon dynamical entropy. This additional assumption is rather weak, since quasihomogeneity of g implies that the limit exists and always has to be greater or equal to one. First we will prove some technical lemma which is similar to [10, Lemma 6.3] for utility functions with asymptotic elasticity smaller than one.

Lemma 3.1. *Let $g \in \mathcal{G}_0$. For $\lambda > 1$ define*

$$U(\lambda) := \liminf_{x \rightarrow 0^+} \frac{\lambda g(x)}{g(\lambda x)}.$$

Then U is nondecreasing and for every $\lambda \geq 1$ $U(\lambda) \geq 1$. If additionally g is differentiable, then the following conditions are equivalent:

- (i) $U(\lambda)$ is greater than one for every $\lambda > 1$,
- (ii) $\limsup_{x \rightarrow 0^+} \frac{xg'(x)}{g(x)} < 1$.

Proof. Since g is quasihomogenic for $1 \leq s \leq t$ we have

$$\frac{tg(x)}{g(tx)} \geq \frac{sg(x)}{g(sx)} \geq 1$$

for every $x \in (0, 1]$ and $U(1) = 1$. This completes the proof of the first part of the theorem. Assume now that g is differentiable. Then it is sufficient to show that the

following conditions are equivalent

$$(i') \quad \exists x_0 > 0 : \forall x \in (0, x_0) \forall \lambda > 1 \quad g(\lambda x) \leq \lambda^\gamma g(x),$$

$$(ii') \quad \exists x_0 > 0 : \forall x \in (0, x_0) \quad g'(x) \leq \gamma \frac{g(x)}{x}$$

for some $\gamma \in (0, 1)$. The proof will be similar to the proof of [10, Lemma 6.3]

$(i') \Rightarrow (ii')$ Fix $\gamma \in (0, 1)$ Let

$$F(\lambda) := g(\lambda x), \quad G(\lambda) := \lambda^\gamma g(x).$$

Then

$$g'(x) = \frac{F'(1)}{x} \leq \frac{G'(1)}{x} = \frac{\gamma g(x)}{x}.$$

$(ii') \Rightarrow (i')$ Fix $\gamma \in (0, 1)$. Functions F and G are differentiable, $F(1) = G(1)$ and

$$F'(1) = xg'(x) < \gamma g(x) = G'(1).$$

Therefore $F(\lambda) < G(\lambda)$ for $\lambda \in (1, 1+\varepsilon)$ for some $\varepsilon > 0$. To show that $F(\lambda) < G(\lambda)$ for every $\lambda > 1$ let $\bar{\lambda} = \inf\{\lambda > 1 : F(\lambda) = G(\lambda)\}$ and suppose that $\bar{\lambda} < \infty$. Note that we must have $F'(\bar{\lambda}) \geq G'(\bar{\lambda})$ which leads to a contradiction, since from (ii') we have

$$F'(\bar{\lambda}) = xg'(\bar{\lambda}x) < \frac{\gamma}{\bar{\lambda}}g(\bar{\lambda}x) = \frac{\gamma}{\bar{\lambda}}F(\bar{\lambda}) = \frac{\gamma}{\bar{\lambda}}G(\bar{\lambda}) = G'(\bar{\lambda}).$$

□

The main result of this section is the following theorem:

Theorem 3.1. *For every differentiable $g \in \mathcal{G}_0^\infty$ for which*

$$(4) \quad \limsup_{x \rightarrow 0^+} \frac{xg'(x)}{g(x)} < 1,$$

and every $\gamma \geq 0$ there exists a process $(X, \Sigma, \mu, T, \mathcal{P})$, such that

$$h(g, \mathcal{P}) = \gamma.$$

We will provide a construction of the process with a given entropy. For $\gamma = 0$ it is obvious, we can consider systems with trivial dynamics, i.e. a system consisting of a single fixed point with trivial measure. Then we have $h(g, \mathcal{P}) = 0$ for every function $g \in \mathcal{G}_0$. Suppose now that $\gamma > 0$. Before we will prove the theorem we will discuss a well-known construction (see e.g. [6]), which is sometimes called the standard example. It will allow us to generate systems where we will be able to control the growth of static g -entropies $H(g, \mathcal{P}_n)$ for some partition \mathcal{P} , and therefore to find a process with a given g -entropy. The system which we construct will be a subshift over two symbols.

Construction of a system. We define inductively sequence $(b_n)_{n=1}^\infty$ and the family of blocks $\left((B_{n,i})_{i=1}^{b_n}\right)_{n=1}^\infty$. Let $(e_n)_{n=1}^\infty$ and $(r_n)_{n=1}^\infty$ be given sequences of

integers and $\mathcal{A} := \{0, 1\}$. The sequence $(b_n)_{n=1}^\infty$ is defined inductively by

$$b_0 := 2, \text{ and } b_{n+1} := (b_n)^{e_n} \text{ for } n \geq 0.$$

Blocks $\left((B_{n,i})_{i=1}^{b_n}\right)_{n=1}^\infty$ and $\left((B'_{n,i})_{i=1}^{b_{n+1}}\right)_{n=1}^\infty$ are given by $B_{0,i} = i - 1$, for $i = 0, 1$ and $B'_{0,j}$ as concatenation of e_0 blocks $B_{0,i}$ for $1 \leq j \leq b_1$. Then for $n > 1$ we define block $B_{n+1,i}$ as a concatenation of r_n copies of the block $B'_{n,i}$ for $1 \leq i \leq b_n$, and block $B'_{n,j}$ for $1 \leq j \leq b_{n+1}$ is a concatenation of e_n (possibly different) blocks $B_{n,i}$. Then by h_n and h'_n we will denote length of block $B_{n,i}$ and $B'_{n,i}$ respectively. Therefore

$$h_n = \prod_{i=0}^{n-1} r_i \cdot \prod_{j=0}^{n-1} e_j \quad \text{and} \quad h'_n = \prod_{i=0}^{n-1} r_i \cdot \prod_{j=0}^n e_j.$$

The basic idea is as follows: we want to control growth of different blocks of positive measure. So if the sequence $(e_n)_{n=1}^\infty$ which gives us the number of concatenated different blocks will be constant and equal 2, then the growth of blocks will be controlled by concatenating as often as we need as big number of copies of one block as we need and the number of copies will be given by values of the sequence $(r_n)_{n=1}^\infty$.

It is easy to see that the system which we will get is the subshift (X, σ) consisting of sequences for which for every $s < t$ there exist n and i such that $[x_s, \dots, x_t]$ is a subword of a block $B_{n,i}$. The only invariant measure is a Bernoulli measure given by assigning to each block $B_{n,i}$ measure $\mu(B_{n,i}) = 1/b_n$. Specifically for a given length $m \in (h_{n-1}, h_n)$ we define measure μ on cylindrical sets C_0^{m-1} – we find such an integer $j \in \{1, \dots, 2^{n-1} - 1\}$ that the number of different admissible blocks is equal to $2^{2^{n-1}+j}$, now for a given block $[\omega_0 \dots \omega_{m-1}]$ we define

$$\mu(C_0^{m-1}(\omega_0, \dots, \omega_{m-1})) = \begin{cases} 2^{-2^{n-1}-j}, & \text{if there exists such block } B_{n,i}, \\ & \text{that } [\omega_0, \dots, \omega_{m-1}] \text{ is a subword of } B_{n,i} \\ 0, & \text{otherwise.} \end{cases}$$

Such a measure is well defined and invariant. The partition for which we will be able to get a given value of the dynamical g -entropy is the zero-coordinate partition \mathcal{P}^A restricted to the subshift. Now if we pass to the subsequence $(h_n)_{n=1}^\infty$ we obtain

$$\sum_{A \in \mathcal{P}_{h_n}^A} g(\mu(A)) = b_n \cdot g\left(\frac{1}{b_n}\right).$$

Therefore

$$(5) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{A \in \mathcal{P}_n^A} g(\mu(A)) \geq \limsup_{n \rightarrow \infty} \frac{1}{h_n} \cdot b_n g\left(\frac{1}{b_n}\right),$$

Moreover any block of length h_n is uniquely determined by the block of length h'_{n-1} since it is a concatenation of r_{n-1} blocks of length h'_{n-1} . In turn, such a block is determined by the block of length $h_{n-1} + \alpha_{n-1}$, where by α_{n-1} we denote length of block which uniquely determines block of length h_{n-1} . Repeating this argument $n - 1$ times we get that any block of length h_n is determined by a subword consisting of its first $\sum_{i=0}^{n-1} h_i$ letters.

Therefore

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{A \in \mathcal{P}_n^A} g(\mu(A)) \geq \limsup_{n \rightarrow \infty} b_n g\left(\frac{1}{b_n}\right) \Big/ \sum_{i=0}^{n-1} h_i.$$

We will prove that for functions $g \in \mathcal{G}_0^\infty$, which fullfill assumptions of Theorem 3.1 we have

$$(6) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{A \in \mathcal{P}_n^A} g(\mu(A)) = \limsup_{n \rightarrow \infty} b_n g\left(\frac{1}{b_n}\right) \Big/ \sum_{i=0}^{n-1} h_i.$$

For this purpose we should check what happens in our construction when $m \in (h_{n-1}, \sum_{i=0}^{n-1} h_i)$. Then

$$\frac{1}{m} \sum_{A \in \mathcal{P}_m^A} g(\mu(A)) = \frac{1}{m} \sum_{[\omega_0, \dots, \omega_{m-1}] \in \mathcal{A}^m} g(\mu(C_0^{m-1}(\omega_0 \dots \omega_{m-1}))) = \frac{\varphi(2^{-2^{n-1}-j})}{m}$$

for some $j \in \{1, \dots, 2^{n-1}-1\}$. According to the definition of measure μ the counter of this expression will be piecewise constant – if we denote by $(r_{i_j})_{j=1}^l$ a sequence of elements of the sequence $(r_i)_{i=1}^n$ greater than one, the counter will change every $r_{i_1}, \dots, r_{i_{l-1}}$ or r_{i_l} terms. Since g is quasihomogenic the considered functions φ are decreasing, therefore the counter of the expression will be increasing with respect to j . Denote by $b_j^{(n)}$ the minimum length of blocks (all blocks of the same length), for which we get $2^{2^{n-1}+j}$ different cylinders of positive measure in our construction. We can focus on the subsequence

$$\xi_j^{(n)} := \varphi(2^{-2^{n-1}-j}) \Big/ b_j^{(n)}$$

for $j = 1, \dots, 2^{n-1}$. Moreover $\xi_{j+1}^{(n)} = \varphi(2^{-2^{n-1}-j-1}) \Big/ (b_j^{(n)} + \theta_j)$, where $\theta_j \in \{r_{i_1}, \dots, r_{i_l}\}$ for $j = 1, \dots, 2^{n-1}$. It is easy to see that if we want to obtain equality in (5), it is enough to show that the sequence $(\xi_j^{(n)})_{j=1}^{2^{n-1}}$ is non-decreasing. According to Lemma 3.1, the assumption (4) implies that $U(2) = \liminf_{x \rightarrow 0^+} 2g(x)/g(2x)$ is greater than one. Note that for functions satisfying this

condition, for sufficiently large n we obtain

$$\begin{aligned}
\frac{\xi_{j+1}^{(n)}}{\xi_j^{(n)}} &= \frac{\varphi(2^{-2^{n-1}-j-1})}{\varphi(2^{-2^{n-1}-j})} \cdot \frac{b_j^{(n)}}{b_j^{(n)} + \theta_i} \geq \frac{\varphi(2^{-2^{n-1}-j-1})}{\varphi(2^{-2^{n-1}-j})} \bigg/ \left(1 + \frac{\max_{i=0, \dots, n-1} r_i}{b_j^{(n)}}\right) \\
&\geq \frac{\varphi(2^{-2^{n-1}-j-1})}{\varphi(2^{-2^{n-1}-j})} \bigg/ 1 + \left(\prod_{j=0}^{n-1} r_j\right)^{-1} \\
&\geq \frac{U(2)}{1 + \left(\prod_{j=0}^{n-1} r_j\right)^{-1}} = U(2) \cdot \prod_{j=0}^{n-1} r_j \bigg/ \left(\prod_{j=0}^{n-1} r_j + 1\right).
\end{aligned}$$

Since $U(2) > 1$ there exists an integer N , such that for $n > N$ we have $\prod_{j=0}^{n-1} r_j \bigg/ \left(\prod_{j=0}^{n-1} r_j + 1\right) > 1/U(2)$, which implies that for sufficiently large n , sequence $\left(\xi_j^{(n)}\right)_{j=1}^{2^{n-1}}$ is increasing.

Corollary 3.1. *If $g \in \mathcal{G}_0^\infty$ is such that $U(2) > 1$, then*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{A \in \mathcal{P}_n^A} g(\mu(A)) = \limsup_{n \rightarrow \infty} b_n g\left(\frac{1}{b_n}\right) \bigg/ \sum_{i=0}^{n-1} h_i.$$

To complete the proof of Theorem 3.1 it is sufficient to show that for every $\gamma > 0$ there exists a sequence $(r_n)_{n=1}^\infty$, for which

$$\limsup_{n \rightarrow \infty} \varphi\left(\frac{1}{b_n}\right) \bigg/ \sum_{i=0}^{n-1} h_i = \gamma.$$

We can see that for each positive integer n we have

$$\varphi\left(\frac{1}{b_n}\right) \bigg/ \sum_{i=0}^{n-1} h_i = \frac{2^{2^n} \cdot g(2^{-2^n})}{2r_0 + \dots + 2^n r_0 \cdots r_{n-1}} = \frac{2^{n+1} a_n}{2r_0 + \dots + 2^n r_0 \cdots r_{n-1}},$$

where

$$a_n := \varphi(2^{-2^{n+1}}) / 2^{n+1}.$$

Since $\varphi_\eta(2^{-2^{n+1}}) = 2^{n+1}$, the fact that $g \in \mathcal{G}_0^\infty$ implies

$$\lim_{n \rightarrow \infty} a_n = \infty.$$

Therefore it is sufficient to show the following lemma:

Lemma 3.2. *For every sequence of real numbers $(a_n)_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} a_n = \infty$ and for every $\gamma > 0$ there exist such a sequence of integers $(r_n)_{n=1}^\infty$ that*

$$\limsup_{n \rightarrow \infty} \frac{2^{n+1} a_n}{2r_0 + \dots + 2^n r_0 \cdots r_{n-1}} = \gamma.$$

Proof. Let

$$\gamma_n := \frac{2^{n+1}a_n}{2r_0 + \dots + 2^n r_0 \cdots r_{n-1}}.$$

Without loss of generality we may assume that $\gamma = 1$. We will construct sequence $(r_n)_{n \in \mathbb{N}}$ inductively due to the index of the next moment at which we concatenate multiple copies of a given block.

Step 1. Since the sequence $(a_n)_{n=1}^\infty$ converges to infinity there exists an index N_0 , for which

$$(7) \quad 1 - \frac{1}{N_0} < \frac{a_{N_0}}{1 - 2^{-N_0}} \Big/ \left[\frac{a_{N_0}}{1 - 2^{-N_0}} \right] < 1 + \frac{1}{N_0},$$

where $[x]$ is an integer part of x . We may assume that N_0 belongs to a subsequence $(n_k)_{k=1}^\infty$, on which a sequence defined by $b_n := 2^n a_n / (2^n - 1)$ is increasing. We define

$$R_0 := \left[\frac{a_{N_0}}{1 - 2^{-N_0}} \right].$$

and

$$r_i := \begin{cases} R_0, & \text{for } i = 0 \\ 1, & \text{for } i = 1, 2, \dots, N_0. \end{cases}$$

Then by (7) we obtain that $\gamma_{N_0} \in \left(1 - \frac{1}{N_0}, 1 + \frac{1}{N_0}\right)$ and for $n = 0, \dots, N_0$ we have $\gamma_n \leq \gamma_{N_0}$.

Step 2. Let $m > 0$. Assume that the integers N_0, \dots, N_{m-1} and r_n for $n = 1, \dots, N_{m-1}$ are already defined. Then there exists such an integer $N > N_{m-1}$, that for every $n > N$ we have

$$(8) \quad 1 - \frac{1}{n+1} < \left[\frac{a_n}{R_0 \cdots R_{m-1} (1 - 2^{N_{m-1}-n})} \right] \Big/ \frac{a_n}{R_0 \cdots R_{m-1} (1 - 2^{N_{m-1}-n})}.$$

Moreover the following technical fact, which simple but technical proof we omit is true:

Remark 3.1. *There exists such an integer $N' \in \mathbb{N}$, that for $n > N'$ we have*

$$\sigma_2 < \left[\frac{a_n}{R_0 \cdots R_{m-1} (1 - 2^{N_{m-1}-n})} \right] \leq \frac{a_n}{R_0 \cdots R_{m-1} (1 - 2^{N_{m-1}-n})} < \sigma_1,$$

where

$$\sigma_1 = \frac{n}{n-1} \frac{a_n}{R_0 \cdots R_{m-1} (1 - 2^{N_{m-1}-n})}^{-\alpha_n}, \quad \sigma_2 = \frac{n}{n+1} \frac{a_n}{R_0 \cdots R_{m-1} (1 - 2^{N_{m-1}-n})}^{-\alpha_n}$$

and

$$\alpha_n := \frac{2^{N_0-n} (1 - 2^{-N_0})}{(1 - 2^{N_{m-1}-n}) R_1 \cdots R_{m-1}} + \dots + \frac{2^{N_{m-2}-n} (1 - 2^{N_{m-2}-N_{m-1}})}{1 - 2^{N_{m-1}-n}}.$$

Now let $N_m > \max\{N, N'\}$ be such that N_m belongs to the subsequence $(n_k)_{k=1}^\infty$, for which sequence $b_{n_k} = a_{n_k} / (c2^{N_{m-1}-n_k} + d(1 - 2^{N_{m-1}-n_k}))$ – with $c, d > 0$ – is

increasing (existence of such subsequence is guaranteed by the fact that a sequence $c_n := b_n/a_n$ is bounded and a_n converges to infinity).

Set

$$R_m := \left[\frac{a_{N_m}}{R_0 \cdots R_{m-1} (1 - 2^{N_{m-1}-N_m})} \right]$$

and

$$r_i := \begin{cases} R_m, & \text{for } i = N_{m-1} + 1 \\ 1, & \text{for } i = N_{m-1} + 2, \dots, N_m. \end{cases}$$

Then by Remark 3, after simple calculations we get that

$$\gamma_{N_m} = a_{N_m} / (R_0(1 - 2^{-N_0})2^{N_0-N_m} + \dots + R_0 \cdots R_m (1 - 2^{N_{m-1}-N_m})) \in \left(1 - \frac{1}{N_m}, 1 + \frac{1}{N_m}\right)$$

and for $n = N_{m-1} + 1, \dots, N_m$ we have $\gamma_n < \gamma_{N_m}$. Eventually

$$\limsup_{n \rightarrow \infty} \frac{2^{n+1}a_n}{2r_0 + \dots + 2^n r_0 \cdots r_{n-1}} = \lim_{m \rightarrow \infty} \gamma_{N_m} = 1.$$

□

In the proof of Theorem 3.1 the crucial role played the inequality $U(2) > 1$. We used this condition to prove equality (6). It is easy to see that we can further weaken the assumption – it is sufficient to show that there exists such $\lambda > 1$ that $U(\lambda) > 1$. Then, applying Lemma 3.1 we can repeat the above construction defining:

- an integer $k := \min\{\lambda > 1 \mid U(\lambda) > 1\}$,
- an alphabet $\mathcal{A} := \{0, 1, \dots, k-1\}$ and
- a sequence (e_n) as $e_n := k$ for every integer n .

Corollary 3.2. *For every function $g \in \mathcal{G}_0^\infty$ for which*

$$(9) \quad \lim_{\lambda \rightarrow \infty} \liminf_{x \rightarrow 0^+} \frac{\lambda g(x)}{g(\lambda x)} > 1.$$

and every $\gamma \geq 0$ there exists a process $(X, \Sigma, \mu, T, \mathcal{P})$, such that

$$h(g, \mathcal{P}) = \gamma.$$

Lemma 3.1 implies that the limit in (9) exists. Moreover this condition is not very restrictive – for every $g \in \mathcal{G}_0$ there is $\liminf_{x \rightarrow 0^+} \lambda g(x)/g(\lambda x) \geq 1$. Finally it should be noted that there are functions in \mathcal{G}_0^∞ for which this limit is equal one, i. e. it is the case for the function $g(x) = x(\ln x - 1)^2$.

Note that we needed assumption (9) only to show that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{A \in \mathcal{P}_n^A} g(\mu(A)) \leq \limsup_{n \rightarrow \infty} \frac{1}{\sum_{i=0}^{n-1} h_i} \cdot b_n g\left(\frac{1}{b_n}\right).$$

To prove this inequality even weaker assumptions are sufficient – condition (9) implies monotonicity of the sequence $(\xi_i^{(n)})$ when in fact it would be sufficient to show just that the upper limit in (6) is realized on the blocks of length $\sum_{i=0}^{n-1} h_i$. Those conditions are usually hard to check and in our opinion will not significantly expand our knowledge. Nonetheless we can formulate the following corollary:

Corollary 3.3. *If $g \in \mathcal{G}_0^\infty$, then for every $\gamma \geq 0$ there exists a process $(X, \Sigma, \mu, T, \mathcal{P})$, such that*

$$h(g, \mathcal{P}) \geq \gamma.$$

4. KOLMOGOROV-SINAI ENTROPY LIKE INVARIANT

The basic tool in the ergodic theory is Kolmogorov-Sinai entropy defined as a supremum of Shannon dynamical entropies over all finite partitions:

$$h_\mu(T) = \sup_{\mathcal{P} - \text{finite}} h(T, \mathcal{P}).$$

It is invariant under metric isomorphism. Following the Kolmogorov proposition we take the supremum over all partitions of dynamical g -entropy of a partition. For a given system (X, Σ, μ, T) we define

$$(10) \quad h_\mu(g, T) = \sup_{\mathcal{P} - \text{finite}} h(g, T, \mathcal{P})$$

and call it the **measure-theoretic g -entropy of transformation T with respect to measure μ** .

It is easy to see that it is an isomorphism invariant. Ornstein and Weiss [14] showed the striking result that measure-theoretic entropy is the only finitely observable invariant for the class of all ergodic processes. More precisely – every finitely observable invariant for a class of all ergodic processes is a continuous function of entropy. Of course in the case of $g \in \mathcal{G}_0^0 \cup \mathcal{G}_0^{\text{Sh}}$ by Corollary 2.2 we have

$$h_\mu(g, T) = \lim_{x \rightarrow 0^+} \frac{g(x)}{\eta(x)} \cdot h_\mu(T).$$

We will show that for a wider class of functions, namely for functions for which

$$\text{Cs}(g) = \limsup_{x \rightarrow 0^+} \frac{g(x)}{\eta(x)} < \infty$$

we have

$$h_\mu(g, T) = \text{Cs}(g) \cdot h_\mu(T)$$

for any ergodic transformation T . This shows that the measure-theoretic g -entropy is in fact finitely observable: one might simply compose the entropy estimators [24] with the linear function itself. Our proof will be similar to the proof of [22, Thm 1.1] where Takens and Verbitski showed that for ergodic transformations

supremum over all finite partitions of dynamical Rényi entropies of order $\alpha > 1$ are equal to the measure-theoretic entropy of T with respect to measure μ .

Let us introduce necessary definitions. Let T_i be automorphisms of Lebesgue space (X_i, Σ_i, μ_i) for $i = 1, 2$ respectively. Then we say that T_2 is a **factor** of transformation T_1 , if there exists a homomorphism $\phi: X_1 \mapsto X_2$ such that

$$\phi T_1 = T_2 \phi \quad \mu_1 \text{ a.e. on } X_1.$$

Suppose that T_2 is a factor of T_1 under homomorphism ϕ . Then for an arbitrary finite partition \mathcal{P} of X_2 we have

$$H \left(g, \bigvee_{i=0}^{k-1} T_2^{-i} \mathcal{P} \right) = H \left(g, \bigvee_{i=0}^{k-1} \phi^{-1} T_2^{-i} \mathcal{P} \right) = H \left(g, \bigvee_{i=0}^{k-1} T_1^{-i} \phi^{-1} \mathcal{P} \right).$$

Hence $h(g, T_2, \mathcal{P}) = h(g, T_1, \phi^{-1} \mathcal{P})$. Therefore

$$h_\mu(g, T_2) = \sup_{\mathcal{P} \text{--finite}} h(g, T_2, \mathcal{P}) = \sup_{\mathcal{P} \text{--finite}} h(g, T_1, \phi^{-1} \mathcal{P}) \leq h(g, T_1).$$

This implies the following proposition:

Proposition 4.1. *If T_2 is a factor of T_1 , then for every function $g \in \mathcal{G}_0$*

$$h_\mu(g, T_2) \leq h_\mu(g, T_1).$$

4.1. Measure-theoretic g -entropies for Bernoulli automorphisms. An automorphism T on (X, Σ, μ) is called **Bernoulli automorphism** if it is isomorphic to some Bernoulli shift. The crucial role in the proof of the main theorem of this section (Theorem 4.2) will play a well-known theorem due to Sinai:

Theorem 4.1 (Sinai, [21]). *Let T be an arbitrary ergodic automorphism of some Lebesgue space (X, Σ, μ) . Then each Bernoulli automorphism with $h_\mu(T_1) \leq h_\mu(T)$ is a factor of the automorphism T .*

The following proposition will play a crucial role in our considerations:

Proposition 4.2. *Let T be an arbitrary ergodic automorphism with $h_\mu(T) \geq \log M$ for some integer $M \geq 2$. Then for every $g \in \mathcal{G}_0$*

$$h_\mu(g, T) \geq \text{Cs}(g) \cdot \log M.$$

Proof. Consider a shift σ over all infinite sequences from the alphabet $\mathcal{A} = \{0, 1, \dots, M-1\}$ with the corresponding Bernoulli measure generated by $p_1 = \dots = p_M = \frac{1}{M}$. It is easy to see that $h_\mu(\sigma) = \log M$. From Theorem 4.1 we conclude that σ is a factor of T . Therefore applying formula (3) we obtain

$$h_\mu(g, T) \geq h_\mu(g, \sigma) \geq h(g, \sigma, \mathcal{P}^{\mathcal{A}}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \varphi(M^{-n}) = \log M \cdot \limsup_{n \rightarrow \infty} \frac{\varphi(M^{-n})}{\varphi_\eta(M^{-n})}.$$

Applying Lemma 2.1 completes the proof. \square

4.2. Main theorem. Our goal in this section is the following result:

Theorem 4.2. *Let T be an ergodic automorphism of Lebesgue space (X, Σ, μ) , and $g \in \mathcal{G}_0$ be such that $\text{Cs}(g) \in (0, \infty)$. Then*

$$h_\mu(g, T) = \begin{cases} \text{Cs}(g) \cdot h_\mu(T), & \text{if } h_\mu(T) < \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

If $g \in \mathcal{G}_0^0$, then $h_\mu(g, T) = 0$. If $g \in \mathcal{G}_0$ is such that $\text{Cs}(g) = \infty$ and T has positive measure-theoretic entropy, then $h_\mu(g, T) = \infty$.

We need a few preliminary lemmas.

Lemma 4.1. *If T is an automorphism of the Lebesgue space (X, Σ, μ) , then for every $g \in \mathcal{G}_0$*

$$h_\mu(g, T^m) \leq m h_\mu(g, T).$$

Proof. Let \mathcal{P} be a finite partition. Then

$$\begin{aligned} h \left(g, T^m, \bigvee_{i=0}^{m-1} T^{-i} \mathcal{P} \right) &= \limsup_{k \rightarrow \infty} \frac{1}{k} H \left(g, \bigvee_{j=0}^{k-1} T^{-mj} \left(\bigvee_{i=0}^{m-1} T^{-i} \mathcal{P} \right) \right) \\ &= m \limsup_{k \rightarrow \infty} \frac{1}{km} H(g, \mathcal{P}_{km-1}) \leq m \limsup_{n \rightarrow \infty} \frac{1}{n} H(g, \mathcal{P}_n) \\ &= m h(g, T, \mathcal{P}) \end{aligned}$$

Taking supremum over all finite partitions we obtain the assertion. \square

Next lemma will be just a weaker version of Theorem 4.2.

Lemma 4.2. *If an automorphism T^m of a Lebesgue space (X, Σ, μ) is ergodic for every $m \in \mathbb{N}$, then for every function $g \in \mathcal{G}_0$, such that $\text{Cs}(g) < \infty$ holds*

$$h_\mu(g, T) = \text{Cs}(g) \cdot h_\mu(T).$$

If $g \in \mathcal{G}_0^0$, then $h_\mu(g, T) = 0$. If $g \in \mathcal{G}_0$ is such that $\text{Cs}(g) = \infty$ and T has positive Kolmogorov-Sinai entropy, then $h_\mu(g, T) = \infty$.

Proof. Case of $g \in \mathcal{G}_0^0$ follows from Corollary 2.2. Suppose that there exists such $g \in \mathcal{G}_0 \setminus \mathcal{G}_0^0$ which fullfills assumptions of lemma and for which we have

$$\text{Cs}(g) \cdot h_\mu(T) - h_\mu(g, T) > 0.$$

Then applying Lemma 4.1 to the transformation T^m and using equality $h_\mu(T^m) = m h_\mu(T)$ (see [9, Thm 4.3.16]) we obtain

$$\text{Cs}(g) h_\mu(T^m) - h_\mu(g, T^m) \geq m (\text{Cs}(g) h_\mu(T) - h_\mu(g, T)) \rightarrow \infty \text{ as } m \rightarrow \infty.$$

Therefore for sufficiently large m there exists an integer M for which

$$(11) \quad h_\mu(g, T^m) \leq m h_\mu(g, T) < \text{Cs}(g) \log M \leq m \text{Cs}(g) h_\mu(T) = \text{Cs}(g) h_\mu(T^m).$$

Proposition 4.2 applied to the transformation T^m guarantees that for every $g \in \mathcal{G}_0$ with positive (finite) $\text{Cs}(g)$ we have

$$(12) \quad h_\mu(g, T^m) \geq \text{Cs}(g) \log M.$$

Comparing (11) and (12) we obtain the contradiction, which implies that

$$h_\mu(g, T) = \text{Cs}(g)h_\mu(T).$$

If $\text{Cs}(g) = \infty$ and $h_\mu(T) > 0$ then there exists such integer $m > 0$ that

$$h_\mu(T^m) = mh_\mu(T) > \log M$$

and by Proposition 4.2 and Lemma 4.1

$$h_\mu(g, T) = h_\mu(g, T^m) = \infty$$

which completes the proof. \square

Proof of Theorem 4.2. If $h_\mu(T) = 0$ theorem is true, since for any partition \mathcal{P} we have

$$0 \leq h(g, \mathcal{P}) \leq \text{Cs}(g)h(\mathcal{P}) = 0.$$

Suppose that $0 < h_\mu(T) < \infty$. Automorphism T is ergodic. Therefore it has factor which is a Bernoulli automorphism T' with entropy $h_\mu(T) = h_\mu(T')$. Every Bernoulli automorphism is mixing, so T^m is ergodic for each m . Applying Lemma 4.2 we obtain

$$h_\mu(g, T') = \text{Cs}(g)h_\mu(T') = \text{Cs}(g)h_\mu(T).$$

Since T' is a factor of T , so Proposition 4.1 implies that

$$\text{Cs}(g)h_\mu(T) = \text{Cs}(g)h_\mu(T') = h_\mu(g, T') \leq h_\mu(g, T) \leq \text{Cs}(g)h_\mu(T)$$

which completes the proof of the case of finite $h_\mu(T)$. If $h_\mu(T) = \infty$, then Proposition 4.2 implies that

$$h_\mu(g, T) \geq \text{Cs}(g) \log M$$

for every $M > 0$ and the theorem is proved. \square

4.3. Case of $g \in \mathcal{G}_0^\infty$. We will prove that for every $g \in \mathcal{G}_0^\infty$ and any aperiodic automorphism T the measure-theoretic g -entropy of the transformation T with respect to μ is infinite. Since we omit the assumption of ergodicity we will use different techniques mainly based on the well-known Rokhlin Lemma which guarantees existence of so called Rokhlin towers of a given height covering sufficiently large part of X . Using such towers we will find lower estimations of g -entropy of a process similar to one obtained by Frank Blume in [2], [3], where he proposed, for a given sequence $(a_n)_{n=1}^\infty$ converging to infinity slower than n , a construction of a partition into two sets \mathcal{P} , for which $\lim_{n \rightarrow \infty} H(\mathcal{P}_n)/a_n = \infty$. We will assume that

we have an aperiodic system, i.e. system (X, Σ, μ, T) for which

$$\mu(\{x \in X : \exists n \in \mathbb{N} T^n x = x\}) = 0.$$

If $M_0, \dots, M_{n-1} \subset X$ are pairwise disjoint sets of equal measure, then $\tau = (M_0, M_1, \dots, M_{n-1})$ is called a **tower**. If additionally $M \subset X$ and $M_k = T^k M$ for $k = 1, \dots, n-1$, then τ is called **Rokhlin tower**.⁴ By the same bold letter τ we will denote the set $\bigcup_{k=0}^{n-1} T^k M$. Obviously $\mu(\tau) = n\mu(M)$. Integer n is called the **height** of tower τ . Moreover for $i < j$ we define a subtower

$$\tau_i^j := (T^i M, \dots, T^j M) \quad \text{and} \quad \tau_i^j = \bigcup_{k=i}^j T^k M.$$

In aperiodic systems there exist Rokhlin towers of a given length and covering sufficiently large part of X :

Lemma 4.3 (Rokhlin, [18]). *If T is an aperiodic transformation of Lebesgue space (X, Σ, μ) , then for every $\varepsilon > 0$ and every integer $n \geq 2$ there exists a Rokhlin tower τ of height n with $\mu(\tau) > 1 - \varepsilon$.*

We now give a definition of an independent collection of sets relative to a Rokhlin tower. We will associate to such collections certain partitions, which will be analogous to Bernoulli partitions. Let $\tau = (M, TM, \dots, T^{n-1}M)$ be a Rokhlin tower. We say that $I \in \Sigma$ is **independent** in τ , if

$$\{T^{-k}(I \cap T^k M), M \setminus T^{-k}(I \cap T^k M)\}_{k=0}^{n-1}$$

is a collection of pairwise disjoint partitions of M . In other words for any $i, j \in \{0, \dots, n-1\}$ we have

$$\mu(T^{-j}(I \cap T^j M) \cap T^{-i}(I \cap T^i M)) = \frac{\mu(I \cap T^j M)\mu(I \cap T^i M)}{\mu(M)}.$$

We can assume that

$$\mu(I \cap T^k M) = \frac{\mu(M)}{2} \quad \text{for } k = 0, \dots, n-1.$$

If T is aperiodic, then such collection exists in every tower since every aperiodic system has no atoms and for any nonatomic Lebesgue space (X, Σ, μ) , every measurable set A and each $\alpha \in [0, \mu(A)]$ one can find a set $B \subset A$ of measure α .

In order to show that the measure-theoretic g -entropy is infinite we need a lower bound for the dynamical g -entropy of a given partition. For this purpose we will use Rokhlin towers and we will calculate dynamical g -entropy with respect to a given Rokhlin tower. This leads us to the following definition: Let \mathcal{P} be a finite partition of X and $F \in \Sigma$, then we define the (static) g -entropy of \mathcal{P} restricted to

⁴It is also known as Rokhlin-Halmos or Rokhlin-Kakutani tower.

F as

$$H_F(g, \mathcal{P}) := \sum_{B \in \mathcal{P}} g(\mu(B \cap F)).$$

The following lemma gives us estimation for $H(g, \mathcal{P})$ from below by the value of g -entropy restricted to a subset of X .

Lemma 4.4. *Let $g \in \mathcal{G}_0$. Let \mathcal{P} be a finite partition such that there exists a set $E \in \mathcal{P}$ with $0 < \mu(E) < 1$. If $F \in \Sigma$, then*

$$H(g, \mathcal{P}) \geq H_F(g, \mathcal{P}) - |g'_-(1/2)| - d_{\max},$$

where $d_{\max} := \max_{x, y \in [0, 1]} |g(x) - g(y)|$.

Proof. By the mean value theorem we have

$$g(\mu(A)) - g(\mu(A \cap F)) = g'_-(x_0^A) (\mu(A) - \mu(A \cap F)),$$

for any set of measure smaller or equal to $1/2$, where $x_0^A \in (\mu(A \cap F), \mu(A))$. Concavity of g implies

$$\sum_{\mu(A) \leq 1/2} (g(\mu(A)) - g(\mu(A \cap F))) \geq g'_-(1/2) \sum_{\mu(A) \leq 1/2} \mu(A \setminus F) \geq -|g'_-(1/2)|.$$

Eventually

$$H(g, \mathcal{P}) - H_F(g, \mathcal{P}) + d_{\max} \geq -|g'_-(1/2)|$$

which completes the proof. \square

Now we give an estimation from below for the g -entropy restricted to a given Rokhlin tower. First, by \mathcal{P}^I we will denote a partition into two sets $\{I, X \setminus I\}$, wfor a measurable set I . Then the following lemma is true.

Lemma 4.5. *Let $\tau = (M, TM, \dots, T^{2n-1}M)$ be Rokhlin tower of height $2n$, $I \in \Sigma$ be an independent set in τ . If $g \in \mathcal{G}_0^\infty$ then*

$$H_{\tau_0^{n-1}}(g, \mathcal{P}_n^I) = \frac{\mu(\tau)}{2} \varphi \left(\frac{\mu(\tau)}{2^{n+1}} \right).$$

Proof. Independence of I in τ implies that the partition

$$\mathcal{P}_n^I \cap \tau_0^{n-1}$$

is a partition of τ_0^{n-1} into 2^n sets of equal measure $2^{-n}\mu(\tau_0^{n-1})$. Therefore

$$\begin{aligned} H_{\tau_0^{n-1}}(g, \mathcal{P}_n^I) &= \sum_{A \in \mathcal{P}_n^I} g(A \cap \tau_0^{n-1}) \\ &= 2^n g \left(\frac{\mu(\tau_0^{n-1})}{2^n} \right) = \mu(\tau_0^{n-1}) \varphi \left(\frac{\mu(\tau_0^{n-1})}{2^n} \right) \\ &= \frac{\mu(\tau)}{2} \varphi \left(\frac{\mu(\tau)}{2^{n+1}} \right). \end{aligned}$$

□

We need to show the continuity of $H(g, \mathcal{P}_n)$ with respect to the partition \mathcal{P}_n if n is fixed. First in the space of all partitions of X into m sets, which will be denoted by \mathfrak{B}_m , we consider pseudometric, which once factored to classes of partitions modulo measure zero becomes a metric. For any $\mathcal{P} = \{A_i, i \in \{1, \dots, m\}\}$, $\mathcal{Q} = \{B_i, i \in \{1, \dots, m\}\}$,⁵ we define

$$d(\mathcal{P}, \mathcal{Q}) = \min_{\pi} \sum_{i=1}^m \mu(A_i \triangle B_{\pi(i)})$$

where the minimum runs through all permutations of the set $\{1, \dots, m\}$ and by \triangle we denote a symmetric difference of two sets. Then:

Lemma 4.6. *If $g \in \mathcal{G}_0$ is a continuous function, then $H(g, \cdot)$ is uniformly continuous on (\mathfrak{B}_m, d) .*

Proof. Fix δ and $\mathcal{P}, \mathcal{P}'$ such that $d(\mathcal{P}, \mathcal{P}') = \delta$. Then (after ordering partition \mathcal{P}' in such way that the distance is realised)

$$\delta = \sum_{i=1}^m \mu(A_i \triangle B_i) \geq \sum_{i=1}^m |\mu(A_i) - \mu(B_i)|.$$

Let $\Delta_m := \{(p_1, \dots, p_m) : p_i \geq 0 \text{ and } \sum_{i=1}^m p_i = 1\}$ be a m -dimensional simplex. Continuity of a transformation

$$\Delta_m \ni x \mapsto \sum_{i=1}^m g(x_i) \in \mathbb{R}$$

implies that $H(g, \cdot)$ is continuous on the compact set and therefore it is uniformly continuous on \mathfrak{B}_m .

□

Theorem 4.3. *Let $g \in \mathcal{G}_0^\infty$ and T be an aperiodic automorphism of a Lebesgue space (X, Σ, μ) . Then*

$$h_\mu(g, T) = \infty.$$

Proof. We will prove that for any $M > 0$ there exists a partition $\mathcal{P}^E = \{E, X \setminus E\}$ such that $h(g, \mathcal{P}) \geq M$. We define recursively a sequence of sets $E_n \in \Sigma$. Let

$$E_0 := \emptyset, \quad N_0 := \delta_0 := 1.$$

Let $n > 0$ and assume that we have already defined E_{n-1} , N_{n-1} and δ_{n-1} . Using Lemma 4.6 we can choose $\delta_n > 0$ such that

$$(13) \quad \delta_n < \frac{1}{2} \delta_{n-1}$$

⁵If $k < m$ we can treat any partition of X into k sets as a partition of X into m sets, since it becomes a partition into m sets by adding $m - k$ copies of an empty set.

$$(14) \quad \left| H(g, \mathcal{P}_{N_n}^{E_{n-1}}) - H(g, \mathcal{P}_{N_n}^F) \right| < 1$$

for any $F \in \Sigma$, for which $\mu(E_{n-1} \triangle F) < 2\delta_n$.

Since

$$\lim_{x \rightarrow 0^+} \frac{g(x)}{\eta(x)} = \infty,$$

we can choose such $N_n \in \mathbb{N}$ that

$$(15) \quad \frac{\varphi(\delta_n 2^{-N_n-1})}{\varphi_\eta(\delta_n 2^{-N_n-1})} > \frac{2M}{\delta_n \log 2}.$$

By Lemma 4.3 there exists $M_n \in \Sigma$, such that $\tau_n = (M_n, TM_n, \dots, T^{2N_n-1}M_n)$ is a Rokhlin tower of measure $\mu(\tau_n) = \delta_n$. Let $I_n \subset \tau_n$ be an independent set in τ_n and

$$E_n := (E_{n-1} \setminus \tau_n) \cup I_n.$$

Then

$$\mu(E_{n-1} \triangle E_n) \leq \mu(\tau_n) = \delta_n.$$

for all positive integers n . By (13) we have $\delta_n < 2^{-n}$ and we conclude that $(\mathbf{1}_{E_n})_{n=0}^\infty$ is a Cauchy sequence in $L_1(X)$. Therefore there exist $E \in \Sigma$ such that $\mathbf{1}_{E_n}$ converges to $\mathbf{1}_E$. For this set we have

$$\mu(E_n \triangle E) \leq \sum_{k=n+1}^\infty \mu(E_k \triangle E_{k-1}) \leq \sum_{k=n+1}^\infty \delta_k < 2\delta_{n+1}.$$

Since $E_n \cap \tau_n = I_n$, applying (14) and Lemmas 4.4 and 4.5 we obtain

$$\begin{aligned} H(g, \mathcal{P}_{N_n}^E) &\geq H(g, \mathcal{P}_{N_n}^{E_n}) - 1 \\ &\geq H_{(\tau_n)_0^{N_n-1}}(g, \mathcal{P}_{N_n}^{E_n}) - |g'(1/2)| - d_{\max} - 1 \\ &\geq \left[\frac{\mu(\tau_n) \log 2}{2} (N_n + 1) - \frac{\mu(\tau_n) \log \mu(\tau_n)}{2} \right] \cdot \frac{\varphi(\mu(\tau_n) 2^{-N_n-1})}{\varphi_\eta(\mu(\tau_n) 2^{-N_n-1})} \\ &\quad - |g'(1/2)| - d_{\max} - 1 \\ &\geq \frac{\log 2}{2} \cdot \mu(\tau_n) \cdot (N_n + 1) \cdot \frac{\varphi(\mu(\tau_n) 2^{-N_n-1})}{\varphi_\eta(\mu(\tau_n) 2^{-N_n-1})} - |g'(1/2)| - d_{\max} - 1. \end{aligned}$$

From (15) we obtain that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{H(g, \mathcal{P}_{N_n}^E)}{N_n} &\geq \lim_{n \rightarrow \infty} \frac{\log 2}{2} \cdot \mu(\tau_n) \cdot \frac{N_n + 1}{N_n} \cdot \frac{\varphi(\mu(\tau_n) 2^{-N_n-1})}{\varphi_\eta(\mu(\tau_n) 2^{-N_n-1})} - \frac{|g'(1/2)| + d_{\max}}{N_n} \\ &\geq \frac{\log 2}{2} \lim_{n \rightarrow \infty} \mu(\tau_n) \cdot \frac{N_n + 1}{N_n} \cdot \frac{\varphi(\mu(\tau_n) 2^{-N_n-1})}{\varphi_\eta(\mu(\tau_n) 2^{-N_n-1})} \\ &\geq M \cdot \lim_{n \rightarrow \infty} \frac{N_n + 1}{N_n} = M. \end{aligned}$$

Since M can be arbitrarily large it completes the proof. \square

4.4. Generator theorem counterpart. In the case of $g \in \mathcal{G}_0^\infty$ there is no counterpart of a Kolmogorov-Sinai generator theorem, which says that the measure-theoretic entropy of the transformation T is realised on every generator of the σ -algebra Σ . Let us consider Sturm shifts – shifts which model translations of the circle $\mathbb{T} = [0, 1)$. Let $\beta \in [0, 1)$ and consider the translation $\phi_\beta: [0, 1) \mapsto [0, 1)$ defined by $\phi_\beta(x) = x + \beta \pmod{1}$. Let \mathcal{P} denote the partition of $[0, 1)$ given by $\mathcal{P} = \{[0, \beta), [\beta, 1)\}$. Then we associate a binary sequence to each $t \in [0, 1)$ according to its itinerary relative to \mathcal{P} ; that is we associate to $t \in [0, 1)$ the bi-infinite sequence x defined by $x_i = 0$ if $\phi_\beta^i(t) \in [0, \beta)$ and $x_i = 1$ if $\phi_\beta^i(t) \in [\beta, 1)$. The set of such sequences is not necessarily closed, but it is shift-invariant and so its closure is a shift space called Sturmian shift. If β is irrational, then Sturmian shift is minimal, i.e. there is no proper subshift. Moreover for a minimal Sturmian shift, the number of n -blocks which occur in an infinite shift space is exactly $n + 1$. Therefore for zero-coordinate partition \mathcal{P}^A , which is a finite generator of σ -algebra Σ and for any function $g \in \mathcal{G}_0$ we have

$$H(g, \mathcal{P}_n^A) = \sum_{A \in \mathcal{P}_n^A} g(\mu_S(A)) \leq \varphi\left(\frac{1}{n+1}\right)$$

where μ_S is the unique invariant measure for Sturm shift. Thus,

$$h(g, \mathcal{P}^A) \leq \limsup_{n \rightarrow \infty} \frac{n+1}{n} g\left(\frac{1}{n+1}\right) = 0.$$

On the other hand since it is strictly ergodic (and thus aperiodic) Theorem 4.3 implies that for any function $g \in \mathcal{G}_0^\infty$

$$h_\mu(g, T) = \infty,$$

therefore we have a finite generator, for which the supremum is not attained.

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